A note on exponential-Möbius sums over $\mathbb{F}_q[t]$

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Abstract

In 1991, Baker and Harman proved, under the assumption of the generalized Riemann hypothesis, that

$$\max_{\theta \in [0,1)} \left| \sum_{n \leq x} \mu(n)e(n\theta) \right| \ll \epsilon x^{3/4+\epsilon}. \quad (1)$$

The purpose of this note is to deduce an analogous bound in the context of polynomials over a finite field using Weil’s Riemann Hypothesis for curves over a finite field. Our approach is based on the work of Hayes who studied exponential sums over irreducible polynomials.

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1. Introduction

Let $\mu$ be the Möbius function and write $e(\theta) = e^{2\pi i \theta}$. Baker and Harman [1] proved under the assumption of the generalized Riemann hypothesis that for all $\epsilon > 0$,

$$\max_{\theta \in [0,1)} \left| \sum_{n \leq x} \mu(n)e(n\theta) \right| \ll \epsilon x^{3/4+\epsilon}. \quad (1)$$
It is conjecture that (1) holds for all $\epsilon > 0$ with $\frac{3}{4}$ replaced by $\frac{1}{2}$. The best unconditional result is due to Davenport [3] who showed that for all $A > 0$

$$\max_{\theta \in [0, 1)} \left| \sum_{n \leq x} \mu(n)e(n\theta) \right| \ll_A \frac{x}{(\log x)^A}.$$ 

The purpose of this note is to deduce an analogue of (1) for the polynomial ring $\mathbb{F}_q[t]$. First, let us go through some definitions required to state the result. The function field analogue of the real numbers is the completion of the field of fractions of $\mathbb{F}_q[t]$ with respect to the norm defined by

$$|f/g| = \begin{cases} q^{\deg f - \deg g} & \text{if } f \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

This completion is naturally identified with the ring of formal Laurent series $\mathbb{F}_q((1/t)) = \{ \sum_{i \leq j} x_it^i : x_i \in \mathbb{F}_q, j \in \mathbb{Z} \}$. The norm defined above is extended to $x = \sum_{i \leq j} x_it^i \in \mathbb{F}_q((1/t))$ by setting $|x| = q^j$ where $j$ is the largest index with $x_j \neq 0$. The analogue of the unit interval is $\mathbb{T} := \{ \sum_{i < 0} x_it^i : x_i \in \mathbb{F}_q \}$, and is a subring of $\mathbb{F}_q((1/t))$.

Define the additive character $\psi : \mathbb{F}_q \to \mathbb{C}^\times$ by $\psi(x) = e(\text{tr}(x)/p)$, where $\text{tr} : \mathbb{F}_q \to \mathbb{F}_p$ is the usual trace map and $p$ is the characteristic of $\mathbb{F}_q$. Define also the exponential map $e_q : \mathbb{F}_q((1/t)) \to \mathbb{C}^\times$ by $e_q(x) = \psi(x^{-1})$.

Now let $\mu(f)$ denote the M"obius function on the ring $\mathbb{F}_q[t]$, defined as $(-1)^k$ if $f$ is the product of $k$ distinct irreducibles and 0 otherwise and let $\phi(f)$ be the size of the unit group $(\mathbb{F}_q[t]/(f))^\times$, that is $|f|\prod_{\omega \mid f} (1 - 1/|\omega|)$, where the product is over all monic irreducibles dividing $f$. All sums over polynomials are sums over monic polynomials.

**Theorem 1.** Suppose $n \geq 3$. Then

$$\max_{\theta \in \mathbb{T}} \left| \sum_{\deg f = n} \mu(f)e_q(f\theta) \right| \leq 4q^{\frac{3n+1}{4}} \left( \frac{3\sqrt{3}}{2} \right)^n.$$ 

**Remark.** It follows that for all $\epsilon > 0$ and $q$ large enough with respect to $\epsilon$ we have

$$\max_{\theta \in \mathbb{T}} \left| \sum_{\deg f = n} \mu(f)e_q(f\theta) \right| \leq q^{\left( \frac{3}{4} + \epsilon \right)n}.$$ 

Our proof of Theorem 1 will follow the strategy of Hayes employed in his study of the exponential sum

$$\sum_{\deg \omega = n, \omega \text{ irreducible}} e_q(\omega \theta).$$
Recently, Bienvenu and Lê have independently derived a similar result to Theorem 1 in [2]. Their Theorem 9 corresponds to our Lemma 1 and their Theorem 11 closely resembles our Theorem 1.

2. Lemmas

Let \( \mathbb{F}_q[t]^{\times} \) be the multiplicative monoid of monic polynomials in \( \mathbb{F}_q[t] \). Whilst investigating the distribution of irreducible polynomials over \( \mathbb{F}_q \), Hayes [4] introduced certain congruences classes on \( \mathbb{F}_q[t]^{\times} \) defined as follows. Let \( s \geq 0 \) be an integer and \( g \in \mathbb{F}_q[t] \). We define an equivalence relation \( \mathcal{R}_{s,g} \) on \( \mathbb{F}_q[t]^{\times} \) by

\[
a \equiv b \mod \mathcal{R}_{s,g} \iff g \text{ divides } a - b \quad \text{and} \quad \left| \frac{a}{q^{\deg a}} - \frac{b}{q^{\deg b}} \right| < \frac{1}{q^s}
\]

It is easy to check that this is indeed an equivalence relation and that for all \( c \in \mathbb{F}_q[t]^{\times} \),

\[
a \equiv b \mod \mathcal{R}_{s,g} \Rightarrow ac \equiv bc \mod \mathcal{R}_{s,g}
\]

so we can define the quotient monoid \( \mathbb{F}_q[t]^{\times}/\mathcal{R}_{s,g} \). Hayes showed that an element of \( \mathbb{F}_q[t]^{\times} \) is invertible modulo \( \mathcal{R}_{s,g} \) if and only if it is coprime to \( g \) and that the units of this quotient monoid form an abelian group of order \( q^s \phi(g) \) which we denote \( \mathcal{R}_{s,g}^{\times} = (\mathbb{F}_q[t]^{\times}/\mathcal{R}_{s,g})^{\times} \).

Given a character (group homomorphism) \( \chi : \mathcal{R}_{s,g}^{\times} \to \mathbb{C} \) we can lift this to a character of \( \mathbb{F}_q[t]^{\times} \) by setting \( \chi(f) = 0 \) if \( f \) is not invertible modulo \( \mathcal{R}_{s,g} \). Associated to each such character is the \( L \)-function \( L(u, \chi) \) defined for \( u \in \mathbb{C} \) with \( |u| < 1/q \) by

\[
L(u, \chi) = \sum_{f \in \mathbb{F}_q[t]^{\times}} \chi(f) u^{\deg f} = \prod_{\omega} (1 - \chi(\omega) u^{\deg \omega})^{-1}
\]

where the product is over all monic irreducibles. When \( \chi \) is a non-trivial character it can be shown that \( L(u, \chi) \) is a polynomial which factorises as

\[
L(u, \chi) = \prod_{i=1}^{d(\chi)} (1 - \alpha_i(\chi) u)
\]

for some \( d(\chi) \leq s + \deg g - 1 \) and each \( \alpha_i(\chi) \) satisfies \( |\alpha_i(\chi)| = 1 \) or \( \sqrt{q} \). This follows from Weil’s Riemann Hypothesis and appears to have been first proved by Rhin in [7].

When \( \chi = \chi_0 \) is the trivial character we have

\[
L(u, \chi_0) = \sum_{f \in \mathbb{F}_q[t]^{\times}} u^{\deg f} = \sum_{f \in \mathbb{F}_q[t]^{\times}} u^{\deg f} \prod_{\omega \mid g} (1 - u^{\deg \omega}) = \frac{1}{1 - qu} \prod_{\omega \mid g} (1 - u^{\deg \omega}).
\]
Lemma 1. Let \( \chi \) be a character modulo \( \mathcal{R}^*_{s,g} \) and \( \deg g \leq n/2 \). Then

\[
\left| \sum_{\deg f = n} \mu(f)\chi(f) \right| \leq \begin{cases} \binom{n+s+\deg g-2}{s+\deg g-2}q^{n/2} & \text{if } \chi \neq \chi_0 \\ \binom{n+r-1}{r-1}(q+1) & \text{if } \chi = \chi_0 \end{cases}
\]

where \( r \) is the number of distinct irreducible divisors of \( g \).

Remark. The bound \( \chi_0 \) is smaller than the one for \( \chi \neq \chi_0 \) when \( n \geq 3 \) because \( \deg g \) is an upper bound for \( r \) and for \( n \geq 3 \)

\[
(q+1) \binom{n + \deg g - 1}{n} \leq \binom{n + \deg g - 2}{n} q^{n/2}.
\]

Proof. Suppose first that \( \chi \neq \chi_0 \). Then

\[
\sum_f \chi(f)\mu(f)u^{\deg f} = L(u, \chi)^{-1} = \prod_{i=1}^{d(\chi)} (1 - \alpha_i(\chi) u)^{-1}
\]

\[
= \sum_{n \geq 0} \left( \sum_{r_1 + \ldots + r_{d(\chi)} = n} \prod_{0 \leq r_i \leq n} \alpha_i(\chi)^{r_i} \right) u^n.
\]

Comparing coefficients and using the triangle inequality we get

\[
\left| \sum_{\deg f = n} \chi(f)\mu(f) \right| = \left| \sum_{r_1 + \ldots + r_{d(\chi)} = n} \prod_{0 \leq r_i \leq n} \alpha_i(\chi)^{r_i} \right| \leq \binom{n + d(\chi) - 1}{d(\chi) - 1} q^{n/2}
\]

\[
\leq \binom{n + s + \deg g - 2}{s + \deg g - 2} q^{n/2}.
\]

When \( \chi = \chi_0 \) is the principal character

\[
L(u, \chi_0)^{-1} = (1 - qu) \prod_{\omega \mid g} (1 + u^{\deg \omega} + u^{2 \deg \omega} + \ldots).
\]

If we write \( \omega_1, \omega_2, \ldots, \omega_r \) for the distinct irreducible divisors of \( g \) then we get, by equating coefficients again,

\[
\left| \sum_{\deg f = n} \chi_0(f)\mu(f) \right| \leq \sum_{a_i \in \mathbb{Z}_{\geq 0}} 1 + q \sum_{\sum_{1 \leq i \leq r} a_i \deg \omega_i = n} 1
\]

\[
= \sum_{a_i \in \mathbb{Z}_{\geq 0}} 1 + q \sum_{\sum_{1 \leq i \leq r} a_i \deg \omega_i = n-1} 1
\]
\[ \sum_{i=1}^{r} b_i \in \mathbb{Z}_{\geq 0} \sum_{1 \leq i \leq r} b_i = n \]

\[ = (q + 1) \left( \frac{n + r - 1}{r - 1} \right). \square \]

**Lemma 2.** For each \( \theta \in \mathbb{T} \) there exist unique coprime polynomials \( a, g \in \mathbb{F}_q[t] \) with \( g \) monic and \( \deg a < \deg g \leq n/2 \) such that

\[ \left| \theta - \frac{a}{g} \right| < \frac{1}{q^{\frac{n}{2} + \deg g}}. \]

**Proof.** See Lemma 3 from [6]. \( \square \)

**Lemma 3.** Let \( \theta \in \mathbb{T} \) and let \( a, g \) be the unique polynomials defined as in Lemma 2 with respect to \( \theta \) and \( n \). Set \( s = n - \left[ \frac{n}{2} \right] - \deg g \). For any \( f_1, f_2 \in \mathbb{F}_q[t] \times \) of degree \( n \) such that \( f_1 \equiv f_2 \mod \mathcal{R}_{s,g} \) we have

\[ e_q(f_1 \theta) = e_q(f_2 \theta). \]

**Proof.** See Lemma 5.2 from [5]. \( \square \)

**Lemma 4.** Suppose \( g \in \mathbb{F}_q[t] \) is square-free. Then

\[ \sum_{d | g} \frac{1}{q^{\deg d}} \leq \left( 1 + \frac{\log(\deg g)}{\log q} \right) e. \]

**Proof.** Order the monic irreducibles \( \omega_1, \omega_2, \ldots, \omega_r \) dividing \( g \) and the monic irreducibles \( P_1, \ldots \) in \( \mathbb{F}_q[t] \) in order of degree (and those of the same degree arbitrarily). Let \( \pi(k) \) be the number of monic irreducibles of degree \( k \) and define \( N \) by \( \sum_{\deg P \leq N - 1} \deg P < \deg g \leq \sum_{\deg P \leq N} \deg P \). Then \( g \) has at most \( \sum_{1 \leq k \leq N} \pi(N) \) irreducible factors. Therefore, since \( \deg P_i \leq \deg \omega_i \), we have

\[ \sum_{d | g} \frac{1}{q^{\deg d}} = \prod_{\omega | g} \left( 1 + \frac{1}{q^{\deg \omega}} \right) \leq \prod_{\deg P \leq N} \left( 1 + \frac{1}{q^{\deg P}} \right) = \prod_{1 \leq k \leq N} \left( 1 + \frac{1}{q^k} \right)^{\pi(k)}. \]

Using \( \pi(k) \leq \frac{q^k}{k} \) this is bounded by

\[ \prod_{1 \leq k \leq N} \left( 1 + \frac{1}{q^k} \right)^{\frac{q^k}{k}} \leq \prod_{1 \leq k \leq N} e^{\frac{1}{q^k}} \leq e^{1 + \log N} = Ne. \]

Now we bound \( N \) in terms of \( \deg g \) as follows
\[ \deg g > \sum_{\deg P \leq N-1} \deg P = \sum_{1 \leq k \leq N-1} \pi(k)k \geq \sum_{k \mid N-1} \pi(k)k = q^{N-1} \]

by the prime number theorem in \( \mathbb{F}_q[t] \). This gives \( N \leq 1 + \frac{\log(\deg g)}{\log q} \) which completes the proof of the Lemma. \( \square \)

3. Proof of Theorem 1

Let \( \theta \in \mathbb{T} \) and choose \( g \) and \( s \) as in Lemma 3. We start by giving an explicit description of a set a representatives for the equivalence relation \( \mathcal{R}_{s,g} \). It is not hard to show that

\[ \mathcal{S}_{s,g} = \{ t^{\frac{\deg b_1}{2}} gb_1 + b_2 \mid \deg b_1 = s, b_1 \text{monic}, \deg b_2 < \deg g \} \]

is such a set. Furthermore,

\[ \mathcal{S}_{s,g}^* = \{ t^{\frac{\deg b_1}{2}} gb_1 + b_2 \mid \deg b_1 = s, b_1 \text{monic}, \deg b_2 < \deg g, (b_2, g) = 1 \} \]

defines a set of reduced representatives modulo \( \mathcal{R}_{s,g} \). See [5] Lemma 7.1 for details.

Then by Lemma 3 and the orthogonality of characters modulo \( \mathcal{R}_{s,g}^* \) we can write

\[
\sum_{\deg f = n} \mu(f) \epsilon_q(f^\theta) = \sum_{b \in \mathcal{S}_{s,g}} \sum_{\deg f = n \mod \mathcal{R}_{s,g}} \mu(f) \epsilon_q(f^\theta) \\
= \sum_{d \mid g} \sum_{b \in \mathcal{S}_{s,g}^* / (g/d,b) = 1} \epsilon_q(bd^\theta) \sum_{\deg f = n \mod \mathcal{R}_{s,g}/d} \mu(fd) \\
= \sum_{d \mid g} \sum_{b \in \mathcal{S}_{s,g}^* / (g/d,b) = 1} \epsilon_q(bd^\theta) \sum_{\deg f = n - \deg d \mod \mathcal{R}_{s,g}/d} \frac{1}{q^s \phi(g/d)} \chi(b) \chi(f) \mu(fd). 
\]

Notice that \( \mu(fd) = \mu(f) \mu(d) \chi_d(f) \) where \( \chi_d(f) \) is the trivial character modulo \( \mathcal{R}_{s,d}^* \).

We can therefore rewrite the above as

\[
= \sum_{d \mid g} \frac{\mu(d)}{q^s \phi(g/d)} \chi \mod \mathcal{R}_{s,g}^* / \left( \sum_{b \in \mathcal{S}_{s,g}^*/d} \chi(b) \right) \left( \sum_{\deg f = n - \deg d} \mu(f) \chi \chi_d(f) \right). 
\]

Now \( \chi \) is a character modulo \( \mathcal{R}_{s,g}^* \) and \( \chi_d \) is a character modulo \( \mathcal{R}_{s,d}^* \). Therefore, \( \chi \chi_d \) is a character modulo \( \mathcal{R}_{s,g}^* \), and so using the triangle inequality and Lemma 1 we can bound this in absolute value by
\[ q^{n/2} \sum_{d|g, g \text{ square-free}} \frac{1}{q^{s+\deg d/2} \phi(g/d)} \left( \frac{n - \deg d + s + \deg g - 2}{s + \deg g - 2} \right) \]

\[ \times \sum_{\chi \mod \mathcal{R}_{s,g/d}^*} \left| \sum_{b \in \mathcal{S}_{s,g/d}} e_q(bd\theta)\overline{\chi}(b) \right|. \]

We bound the Gauss sum over \( \chi \mod \mathcal{R}_{s,g/d}^* \) in the standard way using the Cauchy–Schwarz inequality and Parseval’s identity as follows

\[ \sum_{\chi \mod \mathcal{R}_{s,g/d}^*} \left| \sum_{b \in \mathcal{S}_{s,g/d}} e_q(bd\theta)\overline{\chi}(b) \right| \leq \left( \sum_{\chi \mod \mathcal{R}_{s,g/d}^*} 1 \sum_{\chi \mod \mathcal{R}_{s,g/d}^*} \left| \sum_{b \in \mathcal{S}_{s,g/d}} e_q(bd\theta)\overline{\chi}(b) \right|^2 \right)^{1/2} \]

\[ = \left( q^s\phi(g/d) \sum_{b_1,b_2 \in \mathcal{S}_{s,g/d}} e_q(d(b_1 - b_2)\theta) \sum_{\chi \mod \mathcal{R}_{s,g}^*} \overline{\chi}(b_1)\chi(b_2) \right)^{1/2} \]

\[ = \left( (q^s\phi(g/d))^2 \sum_{b_1=b_2 \in \mathcal{S}_{s,g/d}} e_q((b_1 - b_2)\theta) \right)^{1/2} \]

\[ = (q^s\phi(g/d))^{3/2}. \]

Recall that \( s + \deg g = n - \left[ \frac{n}{2} \right] \geq n/2 \) so that

\[ \left( \frac{n - \deg d + s + \deg g - 2}{s + \deg g - 2} \right) \leq \left( \frac{2n - \left[ \frac{n}{2} \right] - 2}{n - \left[ \frac{n}{2} \right] - 2} \right). \]

We can bound this binomial coefficient using the fact that for all positive integers \( k \),

\[ \sqrt{2\pi} k^{k + \frac{1}{2}} e^{-k + \frac{1}{12k+1}} < k! < \sqrt{2\pi} k^{k + \frac{1}{2}} e^{-k + \frac{1}{12k}}. \]

This precise form of Stirling’s formula is due to Robbins [8]. It follows that if \( k = \left[ \frac{n}{2} \right] \) then

\[ \left( \frac{2n - \left[ \frac{n}{2} \right] - 2}{n - \left[ \frac{n}{2} \right] - 2} \right) < \left( \frac{3k}{k} \right) < \frac{1}{\sqrt{2\pi}} e^{\frac{3k}{2} - \frac{1}{12k+1} - \frac{1}{12k+1}} \frac{(3k)^{3k + \frac{1}{2}}}{k^{k + \frac{1}{2}} (2k)^{2k + \frac{1}{2}}} < \frac{1}{\sqrt{4\pi k/3}} \left( \frac{3\sqrt{3}}{2} \right)^{2k}. \]

Putting it all together with \( \phi(g/d) \leq q^{\deg g - \deg d} \) and Lemma 4 we get
\[
\left| \sum_{\deg f=n} \mu(f) e_q(f \theta) \right| \leq q^{n/2} \frac{1}{\sqrt{2\pi(n-1)/3}} \left( \frac{3\sqrt{2}}{2} \right)^n \sum_{d|g} \frac{(q^n \phi(g/d))^{1/2}}{q^{d^2/2}} \\
\leq q^{n-\frac{1}{2}\left(1+\log \frac{n}{\log q}\right)} e \frac{(1+\log n)}{\sqrt{2\pi(n-1)/3}} \left( \frac{3\sqrt{2}}{2} \right)^n
\]

and Theorem 1 easily follows after a short numerical calculation.

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